Contour Integration

①

Jordan's Lemma

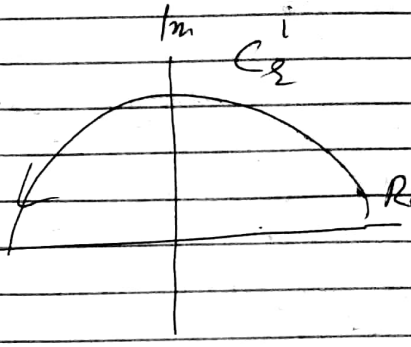
If $t > 0$ and P/q is quotient of two polynomial such that

$$\text{degree}(q) \geq 1 + \text{degree}(P)$$

{ Jordan lemma is helpful to evaluate a contour integral along an infinite open half ~~interval~~ circle as opposed to a closed contour }

, then

$$\lim_{R \to \infty} \int_{C_R^{+i}} e^{izt} \frac{P(z)}{q(z)} dz = 0$$



where C_R^{+i} is an open half circle in the upper half plane

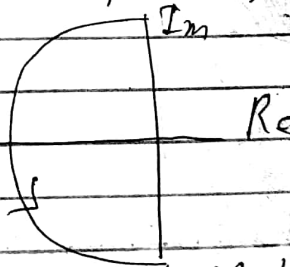
A more useful lemma is by considering Contour in the left half plane

we can change variable such that $s = iz \Rightarrow z = \frac{s}{i}$

and we have for $t > 0$

$$\lim_{R \to \infty} \int_{C_R^-} e^{st} \frac{p(s)}{q(s)} = 0$$

where C_R^- is an open half circle in the left half plane



$p(s)$, $q(s)$ and $p(\frac{s}{i})$ & $q(\frac{s}{i})$

A similar change of variables can be done when considers lower half-plane ($s = -z$, $t < 0$ or with right half plane $s = -iz$, $t < 0$

Improper Integrals involving Rational & Exponential function

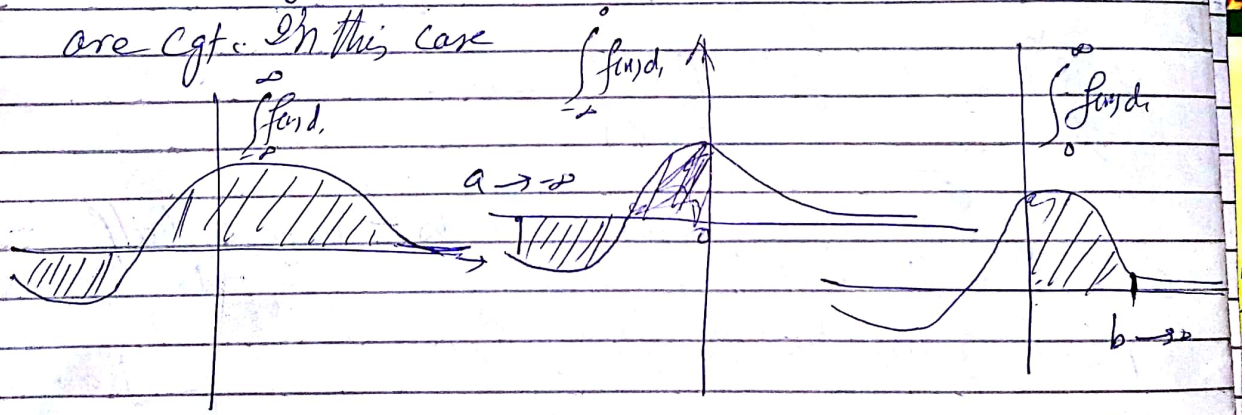
Let a & b be any arbitrary real numbers

Consider the integrals

(1) $\int_{-\infty}^{b} f(x)\,dx$ & $\int_{a}^{\infty} f(x)\,dx$

where in each case f is continuous in the interval of integration and at its finite end point. These are called improper integrals because the interval of integration is infinite. The integral $\int_{a}^{\infty} f(x)\,dx$ is cgt if $\lim_{b\to\infty}\int_{a}^{b} f(x)\,dx$ exists as a finite number. Similarly $\int_{-\infty}^{b} f(x)\,dx$ is convergent if $\lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx$ exists as a finite number.

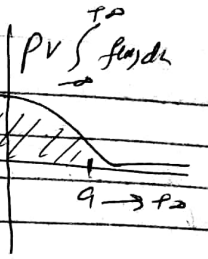
The integral $\int_{-\infty}^{\infty} f(x)\,dx$ is cgt if both $\int_{0}^{\infty} f(x)\,dx$ & $\int_{-\infty}^{0} f(x)\,dx$ are cgt. In this case



Cauchy principal value of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined.

$$P.V\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to\infty}\int_{-a}^{a} f(x)\,dx, \text{ if the limit exists}$$

The Cauchy principal value of an integral may exist even though the integral itself is not cgt. e.g. $\int_{-a}^{a} x\,dx = 0 \quad \forall a$



which shows that $PV\int_{-\infty}^{+\infty} x\,dx = \lim_{a\to\infty}\int_{-a}^{a} x\,dx = 0$ but the integral itself is not cgt since $\int_{0}^{\infty} x\,dx = \infty$.

However when $\int_{-\infty}^{+\infty} f(x)\,dx$ is cgt, then $PV\int_{-\infty}^{+\infty} f(x)\,dx$ exists and the two integrals will be the same. This is because $\lim_{a\to\infty}\int_{-a}^{0} f(x)\,dx$ & $\lim_{a\to\infty}\int_{0}^{a} f(x)\,dx$ both exist

and so $PV\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to\infty}\int_{-a}^{a} f(x)\,dx$

$$= \lim_{a\to\infty}\left(\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\right)$$

$$= \lim_{a\to\infty}\int_{-a}^{0} f(x)\,dx + \lim_{a\to\infty}\int_{0}^{a} f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} f(x)\,dx$$

Because of this fact, we can compute a convergent integral over the real line by computing its principal value which can often be obtained by use of Complex methods and Residue Theorem.

Proposition

Let $\int_A^B f(x)\,dx$ represent an improper integral, where $A \to -\infty$ or $B \to \infty$

(i) If $\int_A^B |f(x)|\,dx$ is cgt, then $\int_A^B f(x)\,dx$ is cgt & where $\left|\int_A^B f(x)\,dx\right| \le \int_A^B |f(x)|\,dx$

(ii) If $|f(x)| \le g(x) \quad \forall x \in]A, B[$ and $\int_A^B g(x)\,dx$ is cgt, then $\int_A^B f(x)\,dx$ is cgt & where $\left|\int_A^B f(x)\,dx\right| \le \int_A^B g(x)\,dx$

We check convergence by this test or by other test like Limit Comparison test to show that the integral is cgt. Once the convergence is determined, we may compute the integral via its principal value.

Example $I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$

Step I # Integrand is ≥ 0 & since a Real line, it is enough that integral outside an finite interval, say $[-1, 1]$ is cgt.

For $|x| \ge 1$, we have

$$\frac{x^2}{x^4+1} \le \frac{x^2}{x^4} \le \frac{1}{x^2}$$

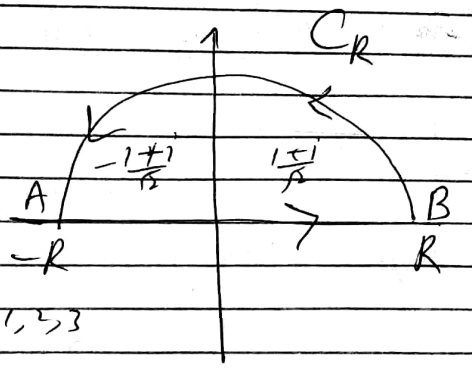
Since $\int_1^{\infty} \frac{1}{x^2}\,dx$ is cgt $\Rightarrow \int_1^{\infty} \frac{x^2}{x^4+1}\,dx$ is cgt & $\int_{-\infty}^{-1} \frac{x^2}{x^4+1}\,dx$ is cgt and so

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = PV\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx$$

Step 2 Consider the function $f(z) = \frac{z^2}{z^4+1}$

We integrate it over a contour that consists interval $[-R, R]$ as a part so that so as to recapture the integral $\int_R^R \frac{x^2}{x^4+1}\,dx$ as a part of contour integral C_R

Choosing the appropriate Contour is not obvious in general. For a rational function such as $\frac{x^2}{x^4+1}$ where the denominator does not vanish on x-axis and degree of denominator is 2 more than the degree of the Numerator a closed semi-circle C_R will work



Step 3 $f(z) = \frac{z^2}{z^4+1}$

$f(z)$ has poles at roots of

$z^4 = -1 \quad i(\pi + 2k\pi)$

$z_k = e^{\frac{i(\pi+2k\pi)}{4}} \quad k = 0, 1, 2, 3$

$z_0 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

$z_1 = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}}$

$z_2 = e^{i5\pi/4} = \cos(\pi+\pi/4) + i\sin(\pi+\pi/4) = -\frac{1-i}{\sqrt{2}}$

$z_3 = e^{i7\pi/4} = \frac{1-i}{\sqrt{2}}$

R 7| Two poles $\frac{1+i}{\sqrt{2}}$, $-\frac{1+i}{\sqrt{2}}$ lie inside C_R

$$\text{Res}(z_0) = \frac{z^2}{\frac{d}{dz}(z^4+1)\big|_{z=z_0}} = \frac{z^2}{4z^3}\Big|_{z=z_0} = \frac{\sqrt{2}}{4(1+i)} = \frac{1-i}{4\sqrt{2}}$$

$$\text{Res}(z_1) = \frac{-1-i}{4\sqrt{2}}$$

By Residue Theorem

$$\int_{C_R} \frac{z^2}{z^4+1}\,dz = 2\pi i\left[\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}}\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{-R} f(z)\,dz + \int_{C_R} f(z)\,dz = \frac{\pi}{\sqrt{2}} \longrightarrow (1)$$

$|\,|z_1| - |z_2|\,| \leq |z_1 - z_2|$

For z on C_R, we have $|z| = R$ $\quad |z^4 - (-1)|$

$$\text{So } \left|\frac{z^2}{z^4+1}\right| \leq \frac{R^2}{|z^4-(-1)|} \leq \frac{R^2}{|\,|z^4| - |-1|\,|} = \frac{R^2}{R^4-1} = M$$

By M.L inequality

$$\left|\int_{C_R} \frac{z^2}{z^4+1}\,dz\right| \leq M \cdot \pi R = \pi R \cdot \frac{R^2}{R^4-1} = \frac{\pi}{R - \frac{1}{R^3}} \to 0$$

as $R \to \infty$

Step-5 Taking limit in (1) when $R \to \infty$

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$$

Note we could have used C_R in the lower half plane. In this case it is easiest to take the orientation of C_R — we in order to coincide with orientation of $[-R, R]$

Proposition 2

Let $f(x) = \frac{P(x)}{q(x)}$ be a rational function with $q(x) \geq 2 +$ degree $P(x)$ & let C_R be arc of a circle centred at 0 with $R > 0$, then

$$\lim_{R \to \infty} \left| \int_{C_R} \frac{P(z)}{q(z)}\,dz \right| = 0$$

Moreover if $q(x)$ has no real roots and $z_1, \ldots, z_N$ denote the poles of $\frac{P(z)}{q(z)}$ in the upper half plane, then

$$\int_{-\infty}^{+\infty} \frac{P(x)}{q(x)}\,dx = 2\pi i \sum_{j=1}^{N} \text{Resi}\left(\frac{P(x)}{q(x)}, z_j\right)$$

Example

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$$

degree of $q(x) = 4 \geq 2 + \text{degree}(P(x)) = 2$.
& denominator has no real roots.
we have.

$$(z^2+1)(z^2+4) = (z+i)(z-i)(z+2i)(z-2i)$$

Then $f(z) = \frac{1}{(z^2+1)(z^2+4)}$ has simple poles $i, 2i$ in upper half-plane

$$\text{Res}(i) = \lim_{z \to i} (z-i) \frac{1}{(z-i)(z+i)(z^2+4)} = \frac{-i}{6}$$

$$\text{Res}(2i) = \frac{i}{12}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} = 2\pi i\left(-\frac{i}{6} + \frac{i}{12}\right) = \frac{\pi}{6}$$

Integral Involving Exponential function

To evaluate integrals involving exponential functions we will use rectangular contours. We consider integral of $\frac{e^{ax}}{e^{bx}+c}$

Example $\int_{-\infty}^{+\infty} \frac{e^{ax}}{e^{bx}+c}$ $a, b, c \neq 0$

Since $c > 0$ and $e^{an} > 0$, the denominator doesn't equal zero. Also condition $a < b$ guarantees that integral is cgt.

Example

Let α be a real no > 1. Prove that

$$\int_0^\infty \frac{e^{\pi x}}{e^{\alpha x}+1}\,dx = \frac{\pi}{\alpha \sin(\pi/\alpha)}$$

Step I : Since $\alpha > 1$, the integral converges

Consider $f(z) = \dfrac{e^{z}}{e^{\alpha z}+1}$

where poles or roots of $e^{\alpha z}+1 = 0$

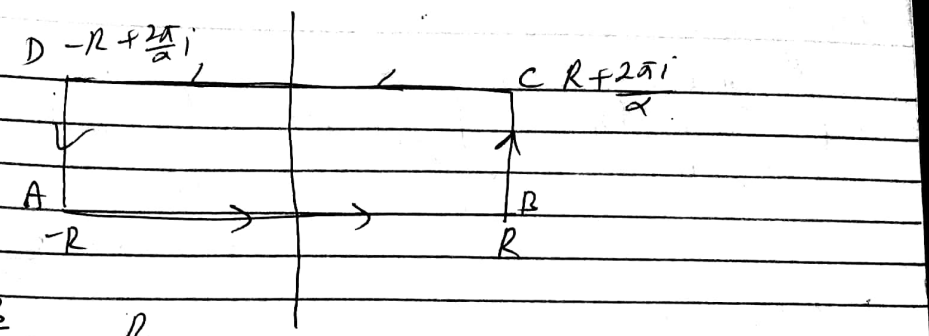
Since the exponential is $2\pi i$ – periodic

$$e^{\alpha z} = -1 = e^{i\pi} \iff \alpha z = i\pi + 2k\pi i$$

$$z_k = (2k+1)\frac{\pi}{\alpha} i \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus $f(z)$ has infinitely many poles at z_k all lying on the imaginary axis

Step-2 Our contour should expand in x-direction so that to cover entire x-axis. To avoid including infinitely many poles in y-axis we will not expand the contour in upper half plane. Instead we use rectangular contour C_R such that C_R expand in horizontal direction, but length in vertical direction remains fixed at 2π. To understand the reason for our choice of vertical length let us consider I_{AB}, I_{CD}



on AB

$$I_{AB} = \int_{-R}^{R} \frac{e^x}{e^{\alpha x}+1}\,dx \quad \because z = x \Rightarrow dz = dx$$

As I_{AB} converges to desired integral as $R \to \infty$

on CD

we have $z = x + i\frac{2\pi}{\alpha}$

$dz = dx$

$$I_{CD} = \int_{R}^{-R} \frac{e^{x+i\frac{2\pi}{\alpha}}}{e^{\alpha(x+\frac{2\pi}{\alpha}i)}+1}\,dx = -e^{\frac{2\pi}{\alpha}i}\int_{-R}^{R} \frac{e^x}{e^{\alpha x}+1}\,dx$$

$$= -e^{\frac{2\pi}{\alpha}i} I_{AB}$$

This last equality explains the choice of vertical sides. They are chosen so that the integral on returning horizontal side CD is equal to constant multiple of integral on AB. From here on is straightforward.

Step 3. We have only one pole at $z_0 = \frac{\pi}{\alpha}i$ inside C_R since $e^{\alpha z}+1$ has simple root, this is simple pole

$$\operatorname{Res}\left(\frac{e^z}{e^{\alpha z}+1}, \frac{\pi}{\alpha}i\right) = \frac{e^{\frac{\pi}{\alpha}i}}{\alpha e^{\alpha\frac{\pi}{\alpha}i}} = -\frac{e^{\frac{\pi}{\alpha}i}}{\alpha}$$

By residue theorem

$$\int_{C_R} \frac{e^z}{e^{\alpha z}+1}\,dz = 2\pi i\left(-\frac{e^{\frac{\pi}{\alpha}i}}{\alpha}\right) = -2\pi i\frac{e^{\frac{\pi}{\alpha}i}}{\alpha}$$

$$\int_{-R}^{R} \frac{e^{\alpha x}}{e^{\alpha x}+1}\,dx + \int_{BC} \frac{e^{\alpha z}}{e^{\alpha z}+1}\,dz + \int_{CD} \frac{e^{\alpha z}}{e^{\alpha z}+1}\,dz + \int_{DA} \frac{e^{\alpha z}}{e^{\alpha z}+1}\,dz$$

$$= 2\pi i \frac{1}{\alpha} \quad \text{--- (A)}$$

Now

$$\int_{BC} \frac{e^{\alpha z}}{e^{\alpha z}+1}\,dz \qquad \left(0 \le y \le \frac{2\pi}{\alpha}\right)$$

$z = R+iy$

$|e^{\alpha z}| = |e^{\alpha R} e^{i\alpha y}| = e^{\alpha R}$

$|e^{\alpha z}+1| \ge |e^{\alpha z}| - 1 = e^{\alpha R} - 1$

$$\Rightarrow \frac{1}{|e^{\alpha z}+1|} \le \frac{1}{e^{\alpha R}-1}$$

So on BC

$$|f(z)| = \left|\frac{e^{\alpha z}}{e^{\alpha z}+1}\right| \le \frac{|e^{R+iy}|}{e^{\alpha R}-1} = \frac{e^R}{e^{\alpha R}-1}$$

$$= \frac{1}{e^{(\alpha-1)R} - e^{-R}}$$

By ML inequality

$$|I_{BC}| = \left|\int_{BC} \frac{e^z}{e^{\alpha z}+1}\,dz\right| \le \frac{l(AB)}{e^{(\alpha-1)R} - e^{-R}}$$

$$= \frac{\frac{2\pi}{\alpha}}{e^{(\alpha-1)R} - e^{-R}} \to 0 \text{ as } R \to \infty$$

Similarly

$|I_{DA}| \to 0$ as $R \to \infty$

④

$$\int_{C_R} \frac{e^z}{e^{2z}+1}\,dz = -2\pi i \sum \frac{e^{\pi/2 i}}{2}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^x}{e^{2x}+1}\,dx + \int_{BC} \frac{e^z}{e^{2z}+1}\,dz + \int_{R}^{-R} \frac{e^z}{e^{2z}+1}\,dz + \int_{DA} \frac{e^z}{e^{2z}+1}\,dz$$

$$= -2\pi i \frac{e^{\pi/2 i}}{2}$$

Taking limit as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx + 0 + \int_{\infty}^{-\infty} \frac{e^{2\pi/2 i}\cdot e^x}{e^{2x}+1}\,dx + 0$$

$$= -2\pi i \frac{e^{\pi/2 i}}{2}$$

$$(1 - e^{\pi/2 i}) \int_{-\infty}^{+\infty} \frac{e^x}{e^{2x}+1}\,dx = -2\pi i \frac{e^{\pi/2 i}}{2}$$

$$\int_{-\infty}^{+\infty} \frac{e^x}{e^{2x}+1}\,dx = \frac{2\pi i}{2\left(e^{\pi/2 i} - e^{\pi/2 i}\right)} = \frac{\pi}{2\delta \pi/2}$$

Note: There are many integrals & related functions that are not computable using semi-circular contours as such as

$$\int_0^{\infty} \frac{dx}{x^3+1}$$

This can be reduced to an integral involving exponential functions by substitution $x = e^t$

(1).

$$\int_0^\infty \frac{1}{x^\alpha+1}\,dx = \frac{\pi}{\alpha \sin \pi/\alpha} \qquad \alpha > 1$$

Integral is convergent, we show that integral converges on $[1, \infty)$, when

$$\frac{1}{x^\alpha+1} \leq \frac{1}{x^\alpha}$$

& the integral converges for $\int_1^\infty \frac{1}{x^\alpha} < \infty$

put $x = e^t \quad dx = e^t\,dt$

$$I = \int_0^\infty \frac{1}{x^\alpha+1}\,dx = \int_{-\infty}^{+\infty} \frac{e^t}{e^{\alpha t}+1}\,dt$$

$$= \int_{-\infty}^{+\infty} \frac{e^x}{e^{\alpha x}+1}\,dx \quad \text{now as above}$$

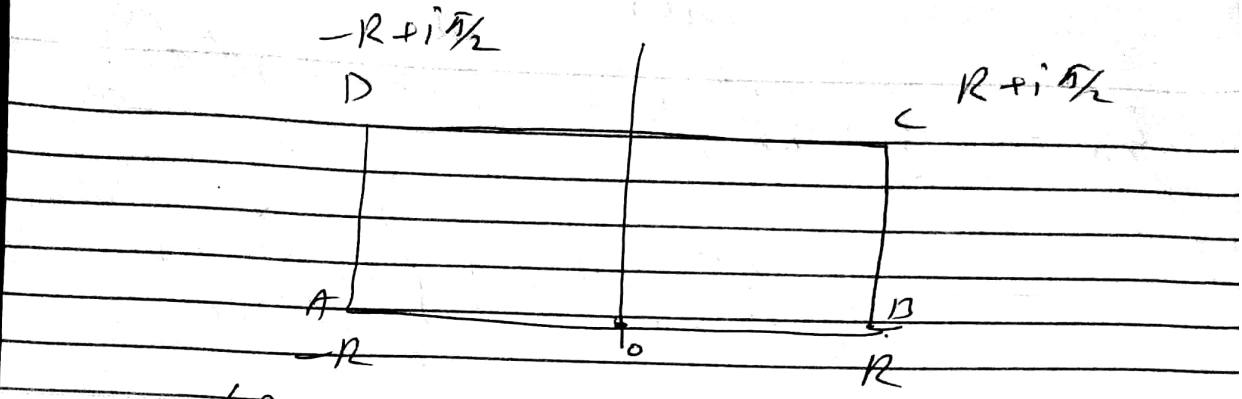
(2). $$\int_0^\infty \frac{\ln x}{x^4+1}\,dx = -\frac{\pi^2}{8\sqrt{2}}$$

let $x = e^t \quad \ln x = t \quad dx = e^t dt$

$$\int_{-\infty}^{+\infty} \frac{t e^t}{e^{4t}+1}\,dt = \int_{-\infty}^{+\infty} \frac{x e^x}{e^{4x}+1}\,dx$$

$$f(z) = \frac{z e^z}{e^{4z}+1}$$

on Rectangular contour C_R



$e^{4z} + 1 = 0 \quad i(\pi + 2k\pi)$

$e^{4z} = -1 = e$

$z_k = \frac{i}{4}(\pi + 2k\pi)$

only $z = i\pi/4$ lie in rectangular contour.

$$\text{Res}\left(\frac{ze^z}{e^{4z}+1}, i\pi/4\right) = \frac{i\pi/4\, e^{i\pi/4}}{4\, e^{i\pi}}$$

$$= \frac{2i\pi(1+i)}{16\sqrt{2}}$$

By Residue th

$$\int_{C_R} f(z)\,dz = 2\pi i\left(\frac{2i\pi(1+i)}{16\sqrt{2}}\right)$$

$$= \frac{\pi^2(1+i)}{8\sqrt{2}}$$

$$I_{AB} + I_{BC} + I_{CD} + I_{DA} = \frac{\pi^2(1+i)}{8\sqrt{2}}$$

$$I_{AB} = \int_{AB} \frac{ze^z}{e^{4z}+1}\,dz = \int_{-R}^{R} \frac{xe^x}{e^{4x}+1}\,dx$$

on CD

$$I_{CD} = \frac{z = R + i\pi/2 \quad dz = dx}{e^{i\pi/2} = i}$$

(14)

$$I_{CD} \to \int_{C_D} \frac{z e^z}{e^{4z}+1}\,dz = \int_{R}^{-R} \frac{(x+i\pi/2)e^x e^{i\pi/2}}{e^{4x}+1}\,dx$$

$$= -i\int_{-R}^{R} \frac{x e^x}{e^{4x}+1}\,dx + \frac{\pi}{2}\int_{-R}^{R} \frac{e^x}{e^{4x}+1}\,dx$$

$$=$$

I_{BC}

Put $z = R+iy \quad (0 \le y \le \pi/2)$

$|z| \le R+y \le R+\pi/2$ also

$$|f(z)| = \left|\frac{z e^z}{e^{4z}+1}\right| = |z|\left|\frac{e^z}{e^{4z}+1}\right|$$

$$\le (R+\pi/2)\frac{e^R}{e^{4R}-1}$$

By M L Inequality

$$|I_{BC}| \le \frac{\pi/2\,(R+\pi/2)\,e^R}{e^{4R}-1} \to 0 \text{ as } R \to \infty$$

Similarly

$I_{DA} \to 0$ as $R \to \infty$

Thm.

$$(1-i)\int_0^{\infty} \frac{x e^x}{e^{4x}+1}\,dx + \pi/2\int_0^{\infty} \frac{e^x}{e^{4x}+1}\,dx = \frac{\pi^2(1+i)}{8\sqrt{2}}$$

Comparing imaginary parts